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Emmanuel Fricain, Javad Mashreghi

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ABSTRACT. We study the convergence of a sequence of finite Blaschke products of a fix order toward a rotation. This would enable us to get a better picture of a characterization theorem for finite Blaschke products.

1. INTRODUCTION

Let $(z_k)_{1\leq k\leq n}$ be a finite sequence in the open unit disc $\mathbb{D}$. Then the rational function

$$B(z)=\gamma\prod_{k=1}^{n}\frac{z_k-z}{1-\bar{z}_k\,z},$$

where γ is a unimodular constant, is called a finite Blaschke product of order n for $\mathbb{D}$ [8]. There are various results characterizing these functions. For example, one of the oldest ones is due to Fatou.

Theorem A (Fatou [5]). *Let f be analytic in the open unit disc $\mathbb{D}$ and suppose that*

$$\lim_{|z|\to 1}|f(z)|=1.$$

Then f is a finite Blaschke product.

For an analytic function $f:\Omega_1\longrightarrow\Omega_2$, the number of solutions of the equation

$$f(z)=w,\qquad (z\in\Omega_1,\ w\in\Omega_2),$$

counting multiplicities, is called the *valence* of f at w and is denoted by $v_f(w)$. It is well-known that a finite Blaschke product of order n has the constant valence n for each $w\in\mathbb{D}$. But, this property in fact characterizes finite Blaschke products of order n.

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Theorem B (Fatou [2, 3, 4], Radó [9]). *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function of constant valence $n \geq 1$. Then f is a finite Blaschke products of order n.*

Our main concern in this paper is the following result of Heins. We remind that a conformal automorphism of $\mathbb{D}$ has the general form

$$T_{a,\gamma}(z) = \gamma\, \frac{a - z}{1 - \bar{a}\, z},$$

where $a \in \mathbb{D}$ and $\gamma \in \mathbb{T}$. Instead of $T_{a,1}$ we simply write T_a. The collection of all such elements is denoted by $\mathrm{Aut}(\mathbb{D})$. The special automorphism

$$\begin{array}{rccc} \rho_\gamma : & \mathbb{D} & \longrightarrow & \mathbb{D} \\ & z & \longmapsto & \gamma z \end{array}$$

is called a rotation. Note that $\rho_\gamma = T_{0,-\gamma}$.

Let us remind two further notions. In the following, when we say that a sequence of functions on $\mathbb{D}$ is convergent, we mean that it converges uniformly on compact subsets of $\mathbb{D}$. If $f : \mathbb{D} \longrightarrow \mathbb{C}$ and $(a_k)_{k\geq 1} \subset \mathbb{D}$, with $\lim_{k\to\infty} |a_k| = 1$, is such that $\lim_{k\to\infty} f(a_k) = L$, then L is called an asymptotic value of f.

Theorem C (Heins [6]). *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function. Then the following assertions are equivalent:*

(i) f is a finite Blaschke product of order ≥ 1;

(ii) if the sequence of automorphisms T_{a_k,γ_k}, $k \geq 1$, tends to a constant of modulus one, and if $g_k = T_{f(a_k\gamma_k)} \circ f \circ T_{a_k,\gamma_k}$ is convergent, then g_k tends to a rotation;

(iii) we have

$$\lim_{|z|\to 1} \frac{(1 - |z|^2)\ |f'(z)|}{1 - |f(z)|^2} = 1;$$

(iv) f has no asymptotic values in $\mathbb{D}$ and the set $\{z \in \mathbb{D} : f'(z) = 0\}$ is finite.

Our contribution is to further clarify the item (ii) above in which it is assumed that the limits of two sequences of analytic functions exist. No doubt the existence of these limits depend on the parameters $a_k \in \mathbb{D}$ and $\gamma_k \in \mathbb{T}$, $k \geq 1$.

It is easy to see that $(T_{a_k,\gamma_k})_{k\geq 1}$ tends to a unimodular constant γ_0 if and only if

$$\lim_{k\to\infty} a_k\, \gamma_k = \gamma_0. \tag{1.1}$$

In the first place, we show that if B is a finite Blaschke product of order ≥ 1 and (1.1) holds, then

$$\big(T_{B(a_k\gamma_k),\,\bar{\gamma}_k} \circ B \circ T_{a_k,\gamma_k}\big)(z) \ \longrightarrow\ \frac{B'(\gamma_0)}{|B'(\gamma_0)|}\, z$$

as $k \longrightarrow +\infty$ and the convergence is uniform on compact subsets of $\mathbb{D}$. See Theorem 3.2. Therefore, the sequence $T_{B(a_k\gamma_k)} \circ B \circ T_{a_k,\gamma_k}$, which was considered in Theorem C, is convergent to a rotation if and only if γ_k is convergent. We refer to the example given at the end to see that this condition cannot be relaxed. In this special situation, if $a_k \longrightarrow \gamma_0$, as $k \longrightarrow +\infty$, then

$$\big(T_{B(a_k)} \circ B \circ T_{a_k} \big)(z) \;\longrightarrow\; \frac{B'(\gamma_0)}{|B'(\gamma_0)|}\, z,$$

uniformly on compact subsets of $\mathbb{D}$. The above observations enable us to rewrite Theorem C in the following form.

Theorem D. *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function. Then the following assertions are equivalent:*

(i) f is a finite Blaschke product of order ≥ 1;
(ii) if $\gamma_k\, a_k \longrightarrow \gamma_0 \in \mathbb{T}$, then $T_{f(a_k\gamma_k),\bar{\gamma}_k} \circ f \circ T_{a_k,\gamma_k}$ tends to a rotation;
(iii) if $a_k \longrightarrow \gamma_0 \in \mathbb{T}$, then $T_{f(a_k)} \circ f \circ T_{a_k}$ tends to a rotation;
(iv) the equality

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\;|f'(z)|}{1-|f(z)|^2} = 1$$

holds;
(v) f has no asymptotic values in $\mathbb{D}$ and the set $\{z \in \mathbb{D} : f'(z) = 0\}$ is finite;
(vi) f has a constant valence on $\mathbb{D}$.

Moreover, if the above conditions hold, then the rotation promised in (ii) *and* (iii) *is ρ_γ, where $\gamma = f'(\gamma_0)/|f'(\gamma_0)|$.*

We will just study the implication $(i) \Longrightarrow (ii)$ in Theorem 3.2. That $(ii) \Longrightarrow (iii)$ is trivial. The implication $(iii) \Longrightarrow (iv)$ is an easy consequence of the formula

$$\frac{(1-|a_k|^2)\; f'(a_k)}{1-|f(a_k)|^2} = \big(T_{f(a_k)} \circ f \circ T_{a_k} \big)'(0),$$

and that, by assumption, the latter tends to a unimodular constant. The more delicate steps $(iv) \Longrightarrow (v) \Longrightarrow (vi)$ and $(vi) \Longrightarrow (i)$ are already taken respectively by Heins and Fatou–Radó.

In the course of proof, we also show that, for a finite Blaschke product B, the zeros of B' which are inside $\mathbb{D}$ are in the hyperbolic convex hull of the zeros of B.

2. HYPERBOLIC CONVEX HULL

The relation $B(z)\, \overline{B(1/\bar{z})} = 1$ shows that

$$B'(z) = 0 \Longleftrightarrow B'(1/\bar{z}) = 0.$$

Hence, to study the singular points of B, it is enough to consider the zeros of B' which are inside the unit disc $\mathbb{D}$. Note that B has no singular points on $\mathbb{T}$, according to (3.1) for instance. Moreover if B has n zero in $\mathbb{D}$, then it is easy to see that B' has $n-1$ zeros in $\mathbb{D}$ (counting with multiplicities).

There are various results about the relations between the zeros of a polynomial P and the zeros of its derivatives. The eldest goes back to Gauss–Lucas [7] which says that the zeros of P' are in the convex hull of the zeros of P. Recently, Cassier–Chalendar [1] established a similar result for Blaschke products: the zeros of B' in $\mathbb{D}$ are in the convex hull of the zeros of B and $\{0\}$. We show that the zeros of B' are in the hyperbolic convex hull of the zeros of B. We do not need to enlarge the zero sets of B by adding $\{0\}$. Moreover, the hyperbolic convex hull of a set is a subset of the Euclidean convex hull of the set and $\{0\}$. In particular, we improve the result obtained in [1].

Let $z_1, z_2 \in \mathbb{D}$. Then the hyperbolic line between z_1 and z_2 is given by

$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{D} \\ t & \longmapsto & \dfrac{z_1 - \frac{z_1-z_2}{1-\overline{z}_1 z_2}\, t}{1 - \overline{z}_1\, \frac{z_1-z_2}{1-\overline{z}_1 z_2}\, t}. \end{array}$$

This representation immediately implies that the three distinct points $z_1, z_2, z_3 \in \mathbb{D}$ are on the same hyperbolic line if and only if

$$\left(\frac{z_1 - z_2}{1-\overline{z}_1 z_2}\right) \Big/ \left(\frac{z_1 - z_3}{1-\overline{z}_1 z_3}\right) \in \mathbb{R}. \tag{2.1}$$

Furthermore, we see that the hyperbolic lines in $\mathbb{D}$ can be parameterized by

$$\gamma \frac{a-z}{1-\overline{a}z} = t, \qquad t \in [-1,1],$$

with $\gamma \in \mathbb{T}$ and $a \in \mathbb{D}$.

Adopting the classical definition from the Euclidean geometry, we say that a set $A \subset \mathbb{D}$ is hyperbolically convex if

$$z_1, z_2 \in A \Longrightarrow \forall t \in [0,1],\ \frac{z_1 - \frac{z_1-z_2}{1-\overline{z}_1 z_2}\, t}{1 - \overline{z}_1\, \frac{z_1-z_2}{1-\overline{z}_1 z_2}\, t} \in A.$$

The hyperbolic convex hull of a set A is the smallest hyperbolic convex set which contains A. This is clearly the intersection of all hyperbolic convex sets that contain A. Of course, if we need the closed hyperbolic convex hull, we must consider the intersection of all closed hyperbolic convex sets that contain A.

In the proof of the following theorem, we need at least two elementary properties of the automorphism

$$T_a(z) = \frac{a-z}{1-\bar{a}\,z}, \qquad (a, z \in \mathbb{D}).$$

Firstly, $T_a \circ T_a = id$. Secondly, if $a \in (-1,1)$, then T_a maps $\mathbb{D}_- = \mathbb{D} \cap \{z : \Im z < 0\}$ into $\mathbb{D}_+ = \mathbb{D} \cap \{z : \Im z > 0\}$. In the same token, we define $\overline{\mathbb{D}}_+ = \mathbb{D} \cap \{z : \Im z \geq 0\}$.

Theorem 2.1. *Let B be a finite Blaschke product. Then the zeros of B' which are inside $\mathbb{D}$ are in the hyperbolic convex hull of the zeros of B.*

Proof. First suppose that all zeros of B are in $\mathbb{D}_+$. Then, taking the logarithmic derivative of B, we obtain

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\bar{z}_k\, z)(z-z_k)},$$

which implies

$$\Im\left(\frac{B'(z)}{B(z)}\right) = \sum_{k=1}^{n} \Im\left(\frac{1-|z_k|^2}{(1-\bar{z}_k\, z)(z-z_k)}\right).$$

In the light of last expression, put

$$\Phi(z) = \frac{1-|a|^2}{(1-\bar{a}\, z)(z-a)},$$

where $a \in \mathbb{D}_+$ is fixed. We need to obtain the image of $\mathbb{D}_-$ under Φ. To do so, we study the image of the boundary of $\partial\mathbb{D}_-$ under Φ. On the lower semicircle

$$\mathbb{T}_- = \{e^{i\theta} : -\pi \leq \theta \leq 0\}$$

we have

$$\Phi(e^{i\theta}) = \frac{1-|a|^2}{(1-\bar{a}\, e^{i\theta})(e^{i\theta}-a)} = \frac{1-|a|^2}{|e^{i\theta}-a|^2}\, e^{-i\theta}.$$

Therefore, $\mathbb{T}_-$ is mapped to an arc in $\mathbb{C}_+$. Moreover, on the interval $t \in (-1,1)$, we have

$$\Phi(t) = \frac{1-|a|^2}{(1-\bar{a}\, t)(t-a)} = \frac{1-|a|^2}{|t-a|^2} \times \frac{t-\bar{a}}{1-\bar{a}\, t} = \frac{1-|a|^2}{|t-a|^2}\, T_t(\bar{a}).$$

Thus $(-1,1)$ is also mapped to an arc in $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$. In short, $\partial\mathbb{D}_-$ is mapped to a closed arc in $\mathbb{C}_+$. Therefore, we deduce that Φ maps $\mathbb{D}_-$ into $\mathbb{C}_+$. Note that 0 is not contained in $\Phi(\mathbb{D}_-)$. This fact implies that B' does not have any zeros in $\mathbb{D}_-$.

By continuity, we can say that if all zeros of B are in $\overline{\mathbb{D}}_+$, then so does all the zeros of B' which are in the disc.

A general automorphism of $\mathbb{D}$ has the form $T_{a,\gamma} = \gamma T_a$, for some $\gamma \in \mathbb{T}$ and $a \in \mathbb{D}$. Let $f = B \circ T_{a,\gamma}$. Then f is also a finite Blaschke product with zeros $T_a(\bar{\gamma}\, z_1)$, $T_a(\bar{\gamma}\, z_2)$, $\ldots$, $T_a(\bar{\gamma}\, z_n)$. Moreover, if we denote the zeros of B' in $\mathbb{D}$ by $w_1, w_2, \ldots, w_{n-1}$, the zeros of f' in $\mathbb{D}$ would be $T_a(\bar{\gamma}\, w_1)$, $T_a(\bar{\gamma}\, w_2), \ldots, T_a(\bar{\gamma}\, w_{n-1})$.

Considering the boundary curves of the hyperbolic convex hull of $z_1, \dots, z_n$, if we choose a and γ such that

$$T_a(\bar\gamma\, z_1),\, T_a(\bar\gamma\, z_2), \dots,\, T_a(\bar\gamma\, z_n) \in \overline{\mathbb{D}}_+,$$

then the preceding observation shows that

$$T_a(\bar\gamma\, w_1),\, T_a(\bar\gamma\, w_2), \dots, T_a(\bar\gamma\, w_{n-1}) \in \overline{\mathbb{D}}_+.$$

Therefore, we see that if the zeros of B are on one side of the hyperbolic line

$$\gamma\, \frac{a-z}{1-\bar a\, z} = t, \qquad t \in [-1,1],$$

then the zeros of B' are also on the same side. The intersection of all such lines gives the hyperbolic convex hull of the zeros of B. $\square$

Remark: The argument above also works for infinite Blaschke products.

Theorem 2.1 is sharp in the sense which is crystalized by the following examples. Let $a, b \in \mathbb{D}$ and put

$$B(z) = \left(\frac{a-z}{1-\bar a\, z} \right)^m \left(\frac{b-z}{1-\bar b\, z} \right)^n .$$

Clearly B' has $m+n-1$ zeros in $\mathbb{D}$ which are a, $m-1$ times, and b, $n-1$ times, and the last one c is given by the equation

$$\frac{m(1-|a|^2)}{(1-\bar a\, z)^2} \left(\frac{a-z}{1-\bar a\, z} \right)^{m-1} \left(\frac{b-z}{1-\bar b\, z} \right)^n + \left(\frac{a-z}{1-\bar a\, z} \right)^m \left(\frac{b-z}{1-\bar b\, z} \right)^{n-1} \frac{n(1-|b|^2)}{(1-\bar b\, z)^2} = 0.$$

Rewrite this equation as

$$\left(\frac{z-a}{1-a\,\bar z} \right) \Big/ \left(\frac{z-b}{1-b\,\bar z} \right) = -\left(\frac{m(1-|a|^2)}{|1-\bar a\, z|^2} \right) \Big/ \left(\frac{n(1-|b|^2)}{|1-\bar b\, z|^2} \right),$$

which, by (2.1), reveals that a, b, c are on the same hyperbolic line. Moreover, as m and n are any arbitrary positive integers, the point c traverses a dense subset of the hyperbolic line segment between a and b.

With more delicate calculations, we can consider examples of the form

$$B(z) = \left(\frac{a-z}{1-\bar a\, z} \right)^m \left(\frac{b-z}{1-\bar b\, z} \right)^n \left(\frac{c-z}{1-\bar c\, z} \right)^p,$$

where $a, b, c \in \mathbb{D}$ and $m, n, p \geq 1$, and observe that the zeros of B', for different values of m, n, p, form a dense subset of the hyperbolic convex hull of a, b, c.

3. Uniform convergence on compact sets

The following result is not stated in its general form. We are content with this version which is enough to obtain our main result that comes afterward.

Lemma 3.1. *Let*

$$B(z) = \gamma \prod_{k=1}^{n} \frac{z_k - z}{1 - \bar{z}_k\, z}$$

and fix any M *such that*

$$\max\{\, |z_1|,\, |z_2|,\, \dots,\, |z_n|\} < M < 1.$$

Then there is a constant $\delta = \delta(M, B) > 0$ *such that, for any two distinct points* z, w *in the annulus* $\{z : M \le |z| \le 1/M\}$, *we have*

$$|z - w| < \delta \Longrightarrow B(z) \neq B(w).$$

Proof. Suppose this is not true. Then, for each $k \ge 1$, there are a_k and b_k such that

$$M \le |a_k|,\, |b_k| \le 1/M, \;\; a_k \neq b_k, \;\; |a_k - b_k| < 1/k,$$

but $B(a_k) = B(b_k)$. Since the closed annulus is compact, $(a_k)_{k\ge1}$ has a convergent subsequence. Without loss of generality, we may assume that $a_k \longrightarrow a$ for some a with $M \le |a| \le 1/M$. This implies $b_k \longrightarrow a$. Therefore, we would have

$$B'(a) = \lim_{k\to\infty} \frac{B(a_k) - B(b_k)}{a_k - b_k} = 0.$$

This is a contradiction. as a matter of fact, by Theorem 2.1, the zeros of B' in $\mathbb{D}$ are in the hyperbolic convex hull of the zeros of B, and those outside $\mathbb{D}$ are the reflection of the zeros inside with respect to $\mathbb{T}$. In other words, B' has no zeros on the annulus. $\square$

There is another way to arrive at a weaker version of Lemma 3.1, which will also suffice for us. On $\mathbb{T}$, the nice formula

$$|B'(e^{i\theta})| = \sum_{k=1}^{n} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} \tag{3.1}$$

holds. Hence, at least, $B'(e^{i\theta}) \neq 0$ for all $e^{i\theta} \in \mathbb{T}$. Thus, by continuity, there is an annulus on which B' has no zeros, and as we explained above, for such a region δ exists.

For the sake of completeness, we remind that if $a_k \in \mathbb{D}$, and $\gamma_k, \gamma_0 \in \mathbb{T}$, are such that

$$a_k \gamma_k \longrightarrow \gamma_0, \qquad \text{as } k \longrightarrow +\infty,$$

then the simple estimation

$$\left|\gamma_0 - \gamma_k \frac{a_k - z}{1 - \bar{a}_k z}\right| \le \frac{2\,|\gamma_0 - a_k\gamma_k|}{1-|z|}, \qquad \text{as } k \longrightarrow +\infty,$$

reveals that

$$T_{a_k,\gamma_k}(z) \longrightarrow \gamma_0 \tag{3.2}$$

uniformly on compact subsets of $\mathbb{D}$.

Theorem 3.2. *Let B be any finite Blaschke product of order ≥ 1. Let $a_k \in \mathbb{D}$ and $\gamma_k, \gamma_0 \in \mathbb{T}$, be such that*

$$\lim_{k\to\infty} a_k\gamma_k = \gamma_0.$$

Then

$$T_{B(a_k\gamma_k),\bar{\gamma}_k} \circ B \circ T_{a_k,\gamma_k} \;\longrightarrow\; \rho_\gamma,$$

where $\gamma = B'(\gamma_0)/|B'(\gamma_0)|$. In other words,

$$\big(T_{B(a_k\gamma_k),\bar{\gamma}_k} \circ B \circ T_{a_k,\gamma_k}\big)(z) \;\longrightarrow\; \frac{B'(\gamma_0)}{|B'(\gamma_0)|}\, z$$

as $k \longrightarrow +\infty$ and the convergence is uniform on compact subsets of $\mathbb{D}$.

Proof. Suppose that B has order n. Then, for each fixed k, the function

$$f_k = T_{B(a_k\gamma_k),\bar{\gamma}_k} \circ B \circ T_{a_k,\gamma_k} \tag{3.3}$$

is also a finite Blaschke product of order n. Its zeros, say

$$z_{k,1},\;\; z_{k,2},\;\; \ldots,\;\; z_{k,n},$$

are the solutions of the equation

$$B\big(T_{a_k,\gamma_k}(z)\big) = B(a_k\gamma_k).$$

Certainly $z = 0$ is a solution. We need to show that the other zeros accumulate to some points on $\mathbb{T}$.

Denote the solutions of $B(w) = B(a_k\gamma_k)$ by

$$w_{k,1},\;\; w_{k,2},\;\; \ldots,\;\; w_{k,n},$$

and put $w_{k,1} = a_k\gamma_k$. As $k \longrightarrow \infty$, we have

$$|w_{k,j}| \longrightarrow 1, \qquad (1 \le j \le n),$$

and, by Lemma 3.1,

$$|w_{k,i} - w_{k,j}| \ge \delta, \qquad (1 \le i \neq j \le n).$$

By the transformation $z = T^{-1}_{a_k,\gamma_k}(w)$, the first solution is mapped to the origin. But the other solutions, thanks to the positive distance between them, are mapped to points close to $\mathbb{T}$. More precisely, we have

$$\lim_{k\to\infty} |z_{k,j}| = \lim_{k\to\infty} |T^{-1}_{a_k,\gamma_k}(w_{k,j})| = 1, \qquad (2 \le j \le n). \tag{3.4}$$

Write

$$f_k(z) = \eta_k \, z \prod_{j=2}^{n} \frac{z_{k,j} - z}{1 - \bar{z}_{k,j}\, z},$$

where $\eta_k \in \mathbb{T}$. This formula shows

$$f_k'(0) = \eta_k \prod_{j=2}^{n} z_{k,j}$$

and (3.3) reveals that

$$\begin{aligned} f_k'(0) &= T'_{B(a_k\gamma_k),\,\bar{\gamma}_k}(B(a_k\gamma_k)) \;\; B'(a_k\gamma_k) \;\; T'_{a_k,\gamma_k}(0) \\ &= \frac{1-|a_k\gamma_k|^2}{1-|B(a_k\gamma_k)|^2} \;\; B'(a_k\gamma_k). \end{aligned}$$

Hence, we obtain the representation

$$f_k(z) = \frac{1-|a_k\gamma_k|^2}{1-|B(a_k\gamma_k)|^2} \;\; B'(a_k\gamma_k) \; \left(\prod_{j=2}^{n} \frac{1}{|z_{k,j}|} \times \prod_{j=2}^{n} \frac{|z_{k,j}|}{z_{k,j}} \; \frac{z_{k,j} - z}{1-\bar{z}_{k,j}\, z} \right) \; z.$$

By (3.2) and (3.4),

$$\left(\prod_{j=2}^{n} \frac{1}{|z_{k,j}|} \times \prod_{j=2}^{n} \frac{|z_{k,j}|}{z_{k,j}} \; \frac{z_{k,j} - z}{1-\bar{z}_{k,j}\, z} \right) \longrightarrow 1,$$

as $k \to \infty$. It is also known that

$$\lim_{z\to\gamma_0} \frac{1-|B(z)|^2}{1-|z|^2} = |B'(\gamma_0)|.$$

We are done. □

Corollary 3.3. *Let B be a finite Blaschke product. Let $a_k \in \mathbb{D}$ and $\gamma_0 \in \mathbb{T}$ be such that*

$$\lim_{k\to\infty} a_k = \gamma_0.$$

Then

$$T_{B(a_k)} \circ B \circ T_{a_k} \;\longrightarrow\; \rho_\gamma, \qquad (k \longrightarrow +\infty),$$

where $\gamma = B'(\gamma_0)/|B'(\gamma_0)|$.

A simple example shows that in Theorem 3.2, $T_{B(a_k\gamma_k),\bar{\gamma}_k}$ cannot be replaced by $T_{B(a_k\gamma_k)}$. In other words, the rotation by $\bar{\gamma}_k$ is needed to obtain the convergence. To see this fact, let a_k be any positive sequence on $[0,1)$ tending to 1. Let $\gamma_k = (-1)^k$, and put $B(z) = z^2$. Then

$$\big(T_{B(a_{2k}\gamma_{2k})} \circ B \circ T_{a_{2k},\gamma_{2k}}\big)(z) \;\longrightarrow\; z$$

while

$$\big(T_{B(a_{2k+1}\gamma_{2k+1})} \circ B \circ T_{a_{2k+1},\gamma_{2k+1}}\big)(z) \;\longrightarrow\; -z.$$

Hence, $T_{B(a_k\gamma_k)} \circ B \circ T_{a_k,\gamma_k}$ is not convergent. Of course, either by Theorem 3.2 or by direct verification,

$$\big(T_{B(a_k\gamma_k),\bar{\gamma}_k} \circ B \circ T_{a_k,\gamma_k}\big)(z) \;\longrightarrow\; z,$$

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Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne.

E-mail address: `fricain@math.univ-lyon1.fr`

Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4.

E-mail address: `Javad.Mashreghi@mat.ulaval.ca`